Economic systems transform inputs — labor, capital, raw materials — into products or outputs. We use a theoretical construct called a production function to summarize the connection between inputs and outputs. Doing this for an entire economy is something of a leap of faith, but it’s an extremely useful device for thinking about economic performance.

The production function

Economic organizations transform inputs (factories, office buildings, machines, labor with a variety of skills, intermediate inputs, and so on) into outputs. Boeing, for example, owns factories, hires workers, buys electricity and avionics, and uses them to produce aircraft. American Express’s credit card business uses computers, buildings, labor, and small amounts of plastic to produce payment services. Pfizer hires scientists, MBAs, and others to develop, produce, and market drugs. McKinsey takes labor and information technology to produce consulting services.

For an economy as a whole, we might think of all the labor and capital used in the economy as producing GDP, the total value of goods and services. A production function is a mathematical relation between inputs and output that makes this idea concrete:

\[ Y = AF(K, L), \]

where \( Y \) is output (real GDP), \( K \) is the quantity of physical capital (plant and equipment) used in production, \( L \) is the quantity of labor, and \( A \) is a measure of the productivity of the economy. More on each of these shortly.

The production function tells us how different amounts of capital and labor may be combined to produce output. We now illustrate a few conditions that we impose on the function \( F \).

- More input leads to more output. Or, in other words, the marginal products of capital and labor are positive. This assumption seems rather uncontroversial to us. In mathematical terms, the function \( F \) increases in both \( K \) and \( L \):

\[ \frac{\partial F}{\partial K} > 0, \quad \frac{\partial F}{\partial L} > 0. \]

Consult the “Mathematics Review” if it is not clear to you why.
• Diminishing marginal products of capital and labor. An increase in labor for given capital leads to increases in output, but it does so at a decreasing rate: the more labor we add, the less additional output we get. This property is illustrated in Figure 1: for a given capital stock $\bar{K}$, increasing labor by $\Delta$ starting from $L_1$ has a larger effect on output than increasing labor by the same amount starting from $L_2$. That is: $AF(\bar{K}, L_1 + \Delta) - AF(\bar{K}, L_1) > AF(\bar{K}, L_2 + \Delta) - AF(\bar{K}, L_2)$. We believe that you’ll be rather comfortable with this assumption: increasing the number of workers in a factory without expanding it will probably increase output, but most probably at a decreasing rate. We assume that the same obtains when we increase capital for given labor. These conditions translate into properties of the second derivatives of the production function:

$$\frac{\partial^2 F}{\partial K^2} < 0, \quad \frac{\partial^2 F}{\partial L^2} < 0.$$  

![Figure 1: The Production Function.](image)

• Constant returns to scale. This property says that if we (say) double all the inputs, the output doubles, too. More formally, if we multiply both inputs by the same number $\lambda > 0$, then we multiply output by the same amount:

$$AF(\lambda K, \lambda L) = \lambda AF(K, L).$$

Thus there is no inherent advantage or disadvantage of size. This assumption has definitely less intuitive appeal than the other two. There is very little hint, in our daily life experience, that it holds true. However, economists have amassed a great deal of empirical evidence in its favor.
These properties are more than we need for most purposes, but we mention them because they play a (sometimes hidden) role in the applications that follow.

Our favorite example of a production function is

\[ Y = AK^\alpha L^{1-\alpha} \]

for a number ("parameter") \( \alpha \) between zero and one. This equation is referred to as the \textit{Cobb-Douglas} version of the production function to commemorate two of the earliest people to use it. (Charles Cobb was a mathematician. Paul Douglas was an economist and later a US senator.) Let’s verify that it satisfies the properties we suggested. First, the marginal products of capital and labor are

\[
\frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha} = \frac{\alpha Y}{K} \\
\frac{\partial Y}{\partial L} = (1 - \alpha)AK^\alpha L^{-\alpha} = (1 - \alpha)Y/N.
\]

Note that both are positive. Second, the marginal products are both decreasing. We show this by differentiating the first derivatives to get second derivatives:

\[
\frac{\partial^2 Y}{\partial K^2} = \alpha(\alpha - 1)AK^{\alpha-2}L^{1-\alpha} \\
\frac{\partial^2 Y}{\partial L^2} = -\alpha(1 - \alpha)AK^\alpha L^{-\alpha-1}.
\]

Note that both are negative. Finally, the function exhibits constant returns to scale. If we multiply both inputs by \( \lambda > 0 \), the result is

\[ A(\lambda K)^\alpha(\lambda L)^{1-\alpha} = A\lambda^\alpha K^\alpha \lambda^{1-\alpha}L^{1-\alpha} = \lambda AK^\alpha L^{1-\alpha}, \]

as needed.

The value of the parameter \( \alpha \) is not made up by economists. If the markets for labor and capital are competitive, so that the providers of one and the other are compensated with their respective marginal products, then \( \alpha \) is the fraction of national income accruing to capital owners, while the complementary fraction \( 1 - \alpha \) is paid to workers. From the NIPA, we learn that capital income accounts for about 1/3 of output. This is true for essentially all countries. For this reason, in most applications we will be assuming \( \alpha = 1/3 \).

\textbf{Productivity}

You may see the word productivity used to mean several different things. The most common measure of productivity is the ratio of output to labor input, which we’ll call the \textit{average product of labor}. This is typically what government agencies mean when they report productivity data. It differs from the \textit{marginal product of labor} for the same reason that average cost differs from marginal cost: the average product is
output produced in average by the units of labor in place, while the marginal product is the increment in output that derives from the addition of one unit to the existing ones. Total factor productivity, or TFP, measures the overall efficiency of the economy in transforming inputs into outputs. Mathematically, the three definitions are

\[
\text{Average Product of Labor} = \frac{Y}{L} \\
\text{Marginal Product of Labor} = \frac{\partial Y}{\partial L} \\
\text{Total Factor Productivity} = \frac{Y}{F(K, L)}.
\]

For the Cobb-Douglas production function they are

\[
\text{Average Product of Labor} = A\left(\frac{K}{L}\right)^\alpha \\
\text{Marginal Product of Labor} = (1 - \alpha)A\left(\frac{K}{L}\right)^\alpha \\
\text{Total Factor Productivity} = A.
\]

Holding \(A\) constant, the first two increase when we increase the ratio of capital to labor. The idea? You can be more productive if you have (say) more equipment to work with. TFP is an attempt to measure productivity independently of the amount of capital each worker has.

**Capital input**

The capital input (or capital stock) \(K\) is the total value of plant and equipment used in production. Ideally, this measure is constructed by evaluating the single assets at their current market price and then summing the values so obtained. Equivalently, we can value single assets at their purchase price and then depreciate their value over time to account for the decline in their productive efficiency. (A sort of economic amortization). The last step is to account for inflation, similarly to what we do with GDP figures. As you can imagine, these exercises are particularly complicated. The main difficulty lies in the evaluation of assets for which there is no active market. This is the case, for example, for old vintages of a given machine.

Fine points:

- How does capital change over time? Typically capital increases with investment (purchased of new plant and equipment) and decreases with depreciation. Mathematically, we might write

\[
K_{t+1} = K_t - \delta_t K_t + I_t,
\]

where \(\delta_t\) is the rate of depreciation between the two dates. Different kinds of capital depreciate at different rates. For example, buildings have longer useful
lives than computers. Estimates of the average depreciation rate range from 6% to 10%.

Wars and disasters can also have an impact. We estimate that the German and Japanese capital stocks declined by about 50% between the start and end of World War II.

- Does land count? In principle maybe it should, but in modern economies land is far less important than plant and equipment. For this reason, we abstract from it when specifying the production function. To the extent it matters, its impact shows up in $A$.

**Labor input**

The ideal measure of labor input is the total number of hours worked. For most countries, the number of people employed is fairly readily available. This is not the case for the amount of hours they put in, in particular for developing countries. For this reason, we will often measure labor with the number of employed.

Labor also differs in quality. CC Sabathia’s skills earn him $23m/year as a pitcher for the Yankees, but most of us have far less skill in the same job. American workers earn more than Mexican workers, in part because their skills are better. There are many skills we might want to measure, but the most important for a country is the level of education of the workforce. In 2010, the average Korean worker had 11.9 years of schooling, and the average Mexican worker had 9.1 years. We know that individuals with more education have higher salaries, on average, so we might guess that Koreans have higher average skills than Mexicans. We take this into account by modifying our production function to include education:

$$Y = AF(K, HL),$$

where $H$ is the average years of education (human capital). We refer to this relation as the *augmented production function*. If we do not adjust $L$ for skill, the effect of $H$ shows up implicitly in $A$. With a Cobb-Douglas production function,

$$Y = AK^\alpha(HL)^{1-\alpha} = (AH^{1-\alpha})K^\alpha L^{1-\alpha},$$

so that $(AH^{1-\alpha})$ is our overall measure of productivity. Of course part of it here is the result of increases in the skill of the workforce.

**Executive summary**

1. A production function links output to inputs.
2. Inputs include physical capital (primarily plant and equipment) and labor.

3. Total Factor Productivity (TFP) is a measure of productive efficiency.

4. Labor varies in quantity (number of people working, numbers of hours) and quality (skill, education).

Review questions

1. Suppose an economy has the production function

\[ Y = AK^{1/4}L^{3/4}. \]

If \( Y = 10 \), \( K = 15 \), and \( L = 5 \), what is total factor productivity \( A \)?
Answer. \( A = Y/(K^{1/4}L^{3/4}) = 1.520. \)

2. Suppose the production function is

\[ Y = 2K^{1/4}L^{3/4} \]

and \( K = L = 1 \). How much output is produced? If we reduced \( L \) by 10\%, how much would \( K \) need to be increased to produce the same output?
Answer. With \( K = L = 1 \), \( Y = 2 \). If \( L \) falls to 0.9, \( K = 1/0.9^3 = 1.372 \) (a 37\% increase in \( K \)). The reason for the difference between the magnitudes in the changes in \( K \) and \( L \) is the difference in their exponents in the production function.

3. Worker 1 has 10 years of education, worker 2 has 15. How much more would you expect worker 2 to earn? Why?
Answer. If \( H = \) years of education, then one hour of worker 2’s time is equivalent to 1.5 (= 15/10) hours of worker 1’s time, so we’d expect her to be paid 50\% more. A more complex answer is that skill may increase in a more complicated way with years of education, and that types of education may differ in their impact on earning power (an MBA may be worth more in this sense than a PhD in cultural anthropology, however interesting the latter may be).

4. Consider the augmented production function

\[ Y = K^{1/3}(HL)^{2/3}. \]

If \( K = 10 \), \( H = 10 \), and \( L = 5 \), what is the average product of labor? How much does the average product increase if \( H \) rises to 12?
Answer. Output is \( Y = 29.24 \) so \( Y/L = 5.85 \). If \( H \) rises to 12, \( Y/L = 6.60. \)

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